Gaussian random-matrix process and universal parametric correlations in complex systems

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We introduce the framework of the Gaussian random-matrix process as an extension of Dyson's Gaussian ensembles and use it to discuss the statistical properties of complex quantum systems that depend on an external parameter. We classify the Gaussian processes according to the short-distance diffusive behavior of their energy levels and demonstrate that all parametric correlation functions become universal upon the appropriate scaling of the parameter. The class of differentiable Gaussian processes is identified as the relevant one for most physical systems. We reproduce the known spectral correlators and compute eigenfunction correlators in their universal form. Numerical evidence from both a chaotic model and weakly disordered model confirms our predictions.

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I. INTRODUCTION

Wigner [1] suggested that the statistical properties of the compound nucleus can be described by a random $N \times N$ Hamiltonian matrix whose elements are independent Gaussian variables. This led to the development of random matrix theory (RMT) [2], which has been successful in describing statistical fluctuations of spectra and eigenfunctions in complex quantum systems drawn from several fields of physics. These include interacting many-body systems such as atomic nuclei [3] and strongly correlated electron models [4], and systems with weak impurity disorder such as quantum wires [5] and quantum dots [6]. There is strong evidence that all systems that are classically chaotic obey the RMT predictions. This is the case even for systems with few degrees of freedom such as billiard models [7] and the hydrogen atom in a strong magnetic field [8].

It has recently been discovered that when these systems are allowed to depend on a parameter (e.g., an external field), the correlations between spectra belonging to different values of the parameter become universal upon an appropriate scaling of the parameter [9–11]. Such universal correlators are the level velocity and the level density correlators. The latter was derived in a closed form using the supersymmetry method [12] for both a specific random-matrix model and for a weakly disordered single-electron system in an external field and was also shown [13] to correspond to the spatiotemporal particle density correlations in the completely integrable Sutherland-Calogero-Moser model [14,15].

More recently, we have introduced the general framework of the Gaussian process (GP) [16] as a natural extension of the Gaussian ensemble that incorporates a parametric dependence and demonstrated that it could be used to obtain the universal form of any correlator. In particular, we computed several universal eigenfunction correlators in the orthogonal case.

In this paper we discuss the theory of the Gaussian process in detail. We show that the different processes are classified according to the diffusive short-distance behavior of their unfolded energy levels $\overline{\Delta\epsilon_i^2} = D\Delta x^\eta$ [16], where $0 < \eta \leq 2$ and D plays the role of the diffusion constant. The levels of GPs in the class $\eta = 1$, which includes Dyson's Brownian motion model [17], execute ordinary diffusion whereas $\eta \neq 1$ processes exhibit anomalous diffusion [18]. We introduce the parameter scaling $x \to \bar{x} = D^{1/\eta}x$ and demonstrate that all correlators become universal functions of $\bar{x}^{\eta/2}$ under this scaling. We show that there are no GPs with $\eta > 2$ and that the $\eta = 2$ processes form the class of differentiable GPs, which is the relevant one for describing most physical systems. Indeed, all the calculations in [10,13] have been performed in $\eta = 2$ models and our scaling procedure reduces in this case to the level-velocity scaling introduced there. The advantage of our approach is that we can calculate (by numerical simulations) the universal form of any parametric correlator that involves the spectra and/or the eigenfunctions through the choice of a simple GP that is easy to construct. The supersymmetry technique, while powerful in deriving analytic expressions, is limited to particular correlators that can be expressed in terms of Green functions.

We discuss different types of GPs: continuous, periodic, and discrete. The latter are particularly important for numerical constructions of the process. For a translation-invariant GP where the correlations depend only on the distance $x - x'$, it is convenient to work in the Fourier space k conjugate to the parameter space x since matrices at different values of k are independent. We define the elementary Gaussian process Φ and use it to describe various constructions of the GP in x and k space. A discrete GP for several values of η is constructed numerically and used to obtain predictions for a number of eigenfunction correlators in their universal form in both the orthogonal and unitary cases. Other correlators are calculated analytically using the supersymmetry method [12]. We confirm these predictions by

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numerical simulations of chaotic and disordered model systems.

The outline of this paper is as follows. In Sec. II we present the GP framework and discuss the different types and classes. Some examples are given. In Sec. III we derive the scaling and the resulting universality of all parametric correlators. In Sec. IV we obtain the universal form of several eigenfunction correlators. Finally, in Secs. V and VI we compute these correlators in two models, a disordered system (the Anderson model) and a chaotic system (the interacting boson model of nuclei) and demonstrate that they take the universal form upon scaling [16]. The latter model also has regular regimes in its parameter space where we observe deviations from universality. Some details of our supersymmetry calculation are provided in the Appendix.

II. THE GAUSSIAN RANDOM-MATRIX PROCESS

A. The continuous Gaussian process

Dyson [19] showed that there are only three possible Gaussian ensembles (GEs) that can characterize a complex physical system, depending on its symmetry. If time-reversal symmetry is conserved, the probability measure is invariant under orthogonal transformations, defining the Gaussian orthogonal ensemble (GOE) of real symmetric matrices. If time-reversal symmetry is broken, the measure is invariant under unitary transformations, defining the Gaussian unitary ensemble (GUE) of complex Hermitian matrices. Finally, if rotational invariance is broken and the total angular momentum is a half-odd integer, the probability measure is invariant under symplectic transformations and the corresponding Gaussian symplectic ensemble (GSE) consists of complex self-dual matrices. The corresponding ensembles are labeled by β, the number of independent components of each matrix element, where β is 1 for the GOE, 2 for the GUE, and 4 for the GSE. Only the cases $\beta = 1, 2$ are treated here.

There are many physical situations in which the above systems depend on some parameter x, e.g., the shape of a billiard or an external magnetic field. The class of statistical properties that are of interest then broadens to include correlations between observables at different parameter values. It is natural to expect that if for all values of x the system belongs to the same symmetry class and its statistical properties are given correctly by the same corresponding GE (e.g., the system is chaotic or weakly disordered at all values of x), these parametric correlators can be obtained from an appropriate generalization of the GE that incorporates this parametric dependence. Such a framework is provided by the GP [16]. A GP is a set of random $N \times N$ matrices $H(x)$ whose elements are distributed at each x according to the appropriate GE with a prescribed correlation among elements at different values of x:

$$\overline{H_{ij}(x)} = 0 ,$$

$$\overline{H_{ij}(x)H_{kl}(x')} = \frac{a^2}{2\beta} f(x,x') g^{(\beta)}_{ij,kl} , \tag{1}$$

where

$$g^{(\beta=1)}_{ij,kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} ,$$
$$g^{(\beta=2)}_{ij,kl} = 2\delta_{il}\delta_{jk} . \tag{2}$$

The matrices $H(x)$ are real symmetric for $\beta = 1$ or complex Hermitian for $\beta = 2$. Notice that f must be symmetric in x and x' and that relations (1) reduce to the standard GE moments when $x = x'$ if $f(x,x) = 1$. We shall be concerned with processes that are translational invariant, i.e., all correlators depend on the distance $x - x'$ rather than on x and x' separately. The process correlation function f (which is symmetric) should therefore satisfy

$$f(x,x') = f(|x - x'|) ,$$
$$f(0) = 1 . \tag{3}$$

A GP is equivalently defined by its probability measure

$$P(H(x))dH(x) \propto \exp\left\{ -\frac{\beta}{2a^2} \int dx dx' \mathrm{Tr}[H(x)K(x,x') \times H(x')] \right\} dH(x) , \tag{4}$$

directly generalizing the GE measure. Since the distribution (4) is Gaussian, the first two moments in (1) completely determine the GP. The relation between the correlation function f and K in (4) will be discussed below. The distribution (4) is invariant under x-independent orthogonal transformations for $\beta = 1$ and unitary transformations for $\beta = 2$. The corresponding GPs are termed the Gaussian orthogonal process (GOP) and the Gaussian unitary process (GUP).

The quadratic form that appears in the exponent of the GP measure (4) can be diagonalized by a Fourier transformation

$$\tilde{H}(k) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} H(x) = \tilde{H}(-k)^\dagger \tag{5}$$

to give

$$P(\tilde{H}(k))d\tilde{H}(k) \propto \exp\left(-\frac{\beta}{2a^2} \int dk \tilde{K}(k) \mathrm{Tr}[\tilde{H}(-k)\tilde{H}(k)] \right) d\tilde{H}(k) , \tag{6}$$

where $\tilde{K}(k)$ is the Fourier transform of $K(x)$

$$K(x - x') = \int \frac{dk}{2\pi} e^{ik(x-x')} \tilde{K}(k) . \tag{7}$$

The process $\tilde{H}(k)$ is δ-function correlated but has a nonuniform variance determined by $\tilde{K}(k)$, which is there-

fore required to be non-negative. Being Gaussian, it is completely specified by its lowest two moments

$$\overline{\tilde{H}_{ij}(k)} = 0 ,$$
$$\overline{\tilde{H}_{ij}(k)\tilde{H}_{kl}(k')} = \frac{a^2}{2\beta}\tilde{f}(k)\delta(k+k')g^{(\beta)}_{ij,kl} , \tag{8}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x-x')$. Comparing with (6) it follows that $K(x-x')$ is just the inverse of $f(x-x')$ in Fourier space,

$$\tilde{K}(k) = \tilde{f}^{-1}(k) . \tag{9}$$

Since $f(x)$ is real and symmetric about $x=0$ and $\tilde{K}(k)$ is non-negative we have

$$\tilde{f}(k) = \tilde{f}(-k) = \tilde{f}^*(k) , \quad \tilde{f}(k) \geq 0 , \tag{10}$$

with $\tilde{K}(k)$ having the exact same properties. Formally (7) can be written as

$$K(x-x') = \tilde{f}^{-1}\left(-i\partial/\partial x\right)\delta(x-x') . \tag{11}$$

Unlike $H(x)$, the matrices $\tilde{H}(k)$ are not Hermitian and hence are not members of any of Dyson's GEs. It is therefore advantageous to define new matrices

$$\tilde{H}^{(+)}(k) = \frac{1}{\sqrt{2}}\left[\tilde{H}(k)+\tilde{H}(-k)\right] ,$$
$$\tilde{H}^{(-)}(k) = \frac{i}{\sqrt{2}}\left[\tilde{H}(k)-\tilde{H}(-k)\right] . \tag{12}$$

As a consequence of (5) the matrices $\tilde{H}^{(\pm)}(k)$ that satisfy $\tilde{H}^{(\pm)}(-k) = \pm\tilde{H}^{(\pm)}(k)$ are real symmetric for $\beta=1$ and complex Hermitian for $\beta=2$. Their distribution law is given by

$$\overline{\tilde{H}^{(\pm)}_{ij}(k)} = 0 ,$$
$$\overline{\tilde{H}^{(\pm)}_{ij}(k)\tilde{H}^{(\pm)}_{kl}(k')} = \frac{a^2}{2\beta}\tilde{f}(k)\delta(k-k')g^{(\beta)}_{ij,kl} ,$$
$$\overline{\tilde{H}^{(\pm)}_{ij}(k)\tilde{H}^{(\mp)}_{kl}(k')} = 0 \tag{13}$$

for $k,k' \geq 0$, indicating that $\tilde{H}^{(\pm)}(k)$ belong to the appropriate GE at each k. This property makes it more convenient to work with $\tilde{H}^{(\pm)}(k)$ rather than $\tilde{H}(k)$ in an actual numerical construction of a GP.

The simplest GP is the δ-function correlated process $f(x-x') = \delta(x-x')$, which consists of a set of independent random matrices $\Phi(x)$ that belong to the appropriate GE and have the same variance. The corresponding probability distribution is given by (4) with $K(x-x') = \delta(x-x')$. The Fourier transform of the process $\tilde{\Phi}(k)$ satisfies (8) with $\tilde{f}(k)=1$ and thus remains δ-function correlated with a uniform variance. Note that this process's correlation function $f(x-x') = \delta(x-x')$ does not satisfy the normalization condition in (3). However, we shall demonstrate now that $\Phi(x)$ is the basic element in the construction of any GP and we term it the elementary Gaussian process (EGP).

$\tilde{H}(k)$ in (8) can be simply related to the Fourier transform $\tilde{\Phi}(k)$ of the EGP through

$$\tilde{H}(k) = \tilde{w}(k)\tilde{\Phi}(k), \tag{14}$$

where $\tilde{w}(k)$ satisfies

$$\tilde{w}(-k) = \tilde{w}^*(k) , \quad |\tilde{w}(k)|^2 = \tilde{f}(k) . \tag{15}$$

Hence the process $H(x)$ can be represented as

$$\tilde{H}(x) = \int \frac{dx}{\sqrt{2\pi}} e^{ikx}\tilde{w}(k)\tilde{\Phi}(k) \tag{16}$$

or alternatively as a convolution of the EGP $\Phi(x)$ with a weight function $w(x)$

$$H(x) = \int dx' w(x-x')\Phi(x') . \tag{17}$$

Here $w(x)$ is the inverse Fourier transform of $\tilde{w}(k)$, which is guaranteed to be real due to the first condition in (15). Note that the second condition there implies a phase ambiguity in $\tilde{w}(k)$,

$$\tilde{w}(k) = |\tilde{f}(k)|^{1/2} e^{i\theta(k)} , \quad \theta(-k) = -\theta(k) . \tag{18}$$

Different choices of $\theta(k)$ would produce different constructions (16) and (17) of the same process $H(x)$ since its probability distribution depends on $\tilde{f}(k)$ alone. Note that it follows from (15) that $f(x) = \int dx' w(x-x')w(-x')$, i.e., $f(x)$ is the convolution of $w(x)$ with $w(-x)$. The representation (17) was introduced in Ref. [20] to study the statistics of avoided crossings in chaotic systems.

We now present several examples of Gaussian processes with various correlation functions. First we consider the Ornstein-Uhlenbeck GP [21] with exponentially decaying correlations

$$f(x) = e^{-\gamma|x|}, \tag{19}$$

where $\gamma > 0$. In Fourier space the correlation becomes a Lorentzian $\tilde{f}(k) = 2\gamma(\gamma^2+k^2)^{-1}$. The choice $\tilde{w}(k) = \sqrt{2\gamma}(\gamma+ik)^{-1}$, which satisfies (15), produces the weight function $w(x) = \sqrt{2\gamma}e^{-\gamma x}\Theta(x)$, yielding the corresponding constructions of the process $H(x)$ from the EGP $\Phi(x)$

$$H(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{\sqrt{2\gamma}e^{ikx}}{\gamma+ik}\tilde{\Phi}(k)$$
$$= \int dx' \sqrt{2\gamma}e^{-\gamma(x-x')}\Theta(x-x')\Phi(x') . \tag{20}$$

An alternative choice of $\tilde{w}(k)$ that differs by a phase from the previous one is $\tilde{w}(k) = \sqrt{2\gamma}(\gamma^2+k^2)^{-1/2}$, producing the weight function $w(x) = \frac{\sqrt{2\gamma}}{\pi}K_0(\gamma|x|)$, where K_0 is the modified Bessel function of order zero. The corresponding GP constructions in this case are

$$H(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{\sqrt{2\gamma}e^{ikx}}{\sqrt{\gamma^2+k^2}}\tilde{\Phi}(k)$$
$$= \int dx' \frac{\sqrt{2\gamma}}{\pi} K_0\left(\gamma|x-x'|\right)\Phi(x') . \tag{21}$$

The function $K(x-x')$ that characterizes the process (19) is easily found from (11) to be $K(x-x') = (1/2\gamma)\left(\gamma^2 - \partial^2/\partial x^2\right)\delta(x-x')$ giving the process probablity distribution

$$P(H(x)) \propto \exp\left\{-\frac{\beta}{2a^2}\int dx \mathrm{Tr}\left[\frac{1}{2\gamma}\left(\frac{dH}{dx}\right)^2 + \frac{\gamma}{2}[H(x)]^2\right]\right\}. \quad (22)$$

The continuous matrix model associated with (22) whose partition function is $\mathcal{Z} = \int D[H(x)]P(H(x))$ was shown in Ref. [22] to be related to the Sutherland-Calogero-Moser class of interacting fermions in one dimension [14,15], which is in turn related to the spectral parametric correlators [13,23].

The second representation in (20) is just the solution to the well-known Brownian motion model of Dyson [17] defined by

$$\dot{H}_{ij} = -\gamma H_{ij} + F_{ij}(t) \quad (23)$$

if we identify $x \leftrightarrow t$ and $\Phi \leftrightarrow F$. γ in (23) plays the role of the friction coefficient and $F_{ij}(t)$ are the random forces

$$\overline{F_{ij}(t)} = 0$$
$$\overline{F_{ij}(t)F_{kl}(t')} = \Gamma\delta(t-t')g^{(\beta)}_{ij,kl}. \quad (24)$$

This model is equivalent to a Fokker-Planck equation for the matrix distribution $P(H,t)$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial H_{ij}}(\gamma H_{ij}P) + \frac{1}{2}\Gamma g^{(\beta)}_{ij,ji}\frac{\partial^2 P}{\partial H_{ij}\partial H^*_{ij}}. \quad (25)$$

For any initial distribution $P(H, t=0)$, as $t \to \infty$ the solution of (25) approaches the GE distribution $P(H) \propto \exp(-\beta \mathrm{Tr}H^2/2a^2)$, which is the equilibrium solution, where we used the fluctuation-dissipation theorem $g^{(\beta)}_{ij,ji}\Gamma/2\gamma = \overline{|H_{ij}|^2}$ to get $\Gamma/\gamma = a^2/\beta$. Therefore, if the initial distribution is chosen to be $P(H)$ we have $P(H,t) = P(H)$ at all t, so $H(t)$ constitutes a GP with $f(t-t') = \exp(-\gamma|t-t'|)$.

As a second example we consider the GP with Gaussian correlations

$$f(x) = e^{-\gamma x^2} \quad (26)$$

for which $\tilde{f}(k) = \sqrt{\frac{\pi}{\gamma}}e^{-k^2/4\gamma}$. The constructions (16) and (17) of $H(x)$ are both given in terms of Gaussian weight functions

$$H(x) = \int \frac{dk}{\sqrt{2\pi}}e^{ikx}\left(\frac{\pi}{\gamma}\right)^{1/4}e^{-k^2/8\gamma}\tilde{\Phi}(k)$$
$$= \int dx'\left(\frac{4\gamma}{\pi}\right)^{1/4}e^{-2\gamma(x-x')^2}\Phi(x), \quad (27)$$

whereas $K(x-x')$ can be formally written as a generalized function $K(x-x') = \sqrt{\gamma/\pi}e^{-\frac{1}{4\gamma}\partial^2/\partial x^2}\delta(x-x')$.

It is shown in Sec. III that the parametric correlators, which are the focus of this work, are determined by the short-distance behavior of the process correlation function

$$f(x) \approx 1 - \kappa|x|^\eta \quad (\eta > 0) \quad (28)$$

rather than by its full functional form. After an appropriate scaling $x \to \bar{x}$ all correlators become universal functions of $|\Delta\bar{x}|^\eta$. It is therefore interesting to construct GPs for different values of η. For $\eta = 1$ and $\eta = 2$ this has been done above in the form of the exponentially correlated (20) and (21) and the Gaussian-correlated (27) processes. A family of GPs with $0 < \eta < 2$ is given by the correlation functions

$$f(x) = \frac{2^{1-\eta/2}}{\Gamma(\frac{\eta}{2})}|x|^{\eta/2}K_{\eta/2}(|x|)$$
$$\approx 1 - \frac{1}{2^\eta}\frac{\Gamma\left(1-\frac{\eta}{2}\right)}{\Gamma\left(1+\frac{\eta}{2}\right)}|x|^\eta, \quad (29)$$

where K_ν is the modified Bessel function of fractional order ν. The Fourier transform of (29) is $\tilde{f}(k) = 2\sqrt{\pi}\frac{\Gamma(\frac{\eta+1}{2})}{\Gamma(\eta/2)}(k^2+1)^{-\frac{\eta+1}{2}}$ and yields the representations

$$H(x) = 2^{1/2}\pi^{1/4}\left[\frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}\right]^{1/2}\int\frac{dk}{\sqrt{2\pi}}\frac{e^{ikx}}{(k^2+1)^{\frac{\eta+1}{4}}}\tilde{\Phi}(k)$$
$$= \left(\frac{2^{3-\eta}}{\pi}\right)^{1/4}\frac{1}{\Gamma(\frac{\eta+3}{4})}\left[\frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}\right]^{1/2}$$
$$\times\int dx'|x-x'|^{\frac{\eta-1}{4}}K_{\frac{\eta-1}{4}}(|x-x'|)\Phi(x'). \quad (30)$$

Note that when $\eta = 1$, $f(x)$ in (29) is just $e^{-|x|}$ and (30) reduces to (21) with $\gamma = 1$. Note also that $f(x)$ defined in (29) is of the general form $f(x) = g_1(x^2) - |x|^\eta g_2(x^2)$, where g_1 and g_2 are analytic functions of x^2, thus $\eta \geq 2$ in (29) gives a GP with $\eta = 2$. In Sec. II D we prove that there are no processes with $\eta > 2$.

B. The periodic Gaussian process

There are many physical situations where the dependence of the Hamiltonian on an external parameter is periodic. This is the case for a metallic ring threaded by a magnetic field whose energies and eigenfunctions are periodic in the flux where the period is the flux quantum $\phi_0 = hc/e$. A random-matrix model that mimics this situation is the periodic GP characterized by $H(x+L) = H(x)$, where L is the period. Its correlation function is also periodic and can be expanded in a Fourier series

$$f(x-x') = \frac{1}{L}\sum_{n=-\infty}^{\infty}e^{i\frac{2\pi n}{L}(x-x')}\tilde{f}_n$$
$$= \frac{\tilde{f}_0}{L} + \frac{2}{L}\sum_{n>0}\tilde{f}_n\cos\frac{2\pi n(x-x')}{L}, \quad (31)$$

with $\tilde{f}_n = \tilde{f}_{-n} \geq 0$. Similarly to (4) and (6) the probability distribution is given by

$$P(H(x)) \propto \exp\left\{-\frac{\beta}{2a^2}\int_0^L \frac{dxdx'}{L^2}\sum_{n=-\infty}^{\infty\prime} e^{i\frac{2\pi n}{L}(x-x')}\tilde{f}_n^{-1} \times \mathrm{Tr}[H(x)H(x')]\right\}$$
$$\propto \exp\left[-\frac{\beta}{2a^2}\sum_{n=-\infty}^{\infty\prime}\tilde{f}_n^{-1}\mathrm{Tr}(\tilde{H}_{-n}\tilde{H}_n)\right], \quad (32)$$

where $\tilde{H}_n = \int dx e^{-i\frac{2\pi n}{L}x}H(x)/\sqrt{L} = \tilde{H}^\dagger_{-n}$ and the prime on the n sum indicates that only $\tilde{f}_n > 0$ should be summed over. The preceding discussion leading to the constructions (16) and (17) can be carried over to the periodic case with minor modifications arising from the discreteness of Fourier space. The construction analogous to (16) is

$$H(x) = \sum_{n=-\infty}^{\infty}\sqrt{\frac{\tilde{f}_n}{L}}e^{i\frac{2\pi k}{L}x}\tilde{\Phi}_n$$
$$= \sqrt{\frac{\tilde{f}_0}{2L}}\tilde{\Phi}_0^{(+)} + \sum_{n>0}\sqrt{\frac{2\tilde{f}_n}{L}}\left[\cos\left(\frac{2\pi nx}{L}\right)\tilde{\Phi}_n^{(+)} + \sin\left(\frac{2\pi nx}{L}\right)\tilde{\Phi}_n^{(-)}\right], \quad (33)$$

where $\tilde{\Phi}_n^{(\pm)}$ is the discrete version of the Fourier-space EGP $\tilde{\Phi}^{(\pm)}(k)$ [see (12)]

$$\overline{\left[\tilde{\Phi}_n^{(\pm)}\right]_{ij}} = 0,$$
$$\overline{\left[\tilde{\Phi}_n^{(\pm)}\right]_{ij}\left[\tilde{\Phi}_{n'}^{(\pm)}\right]_{kl}} = \frac{a^2}{2\beta}\delta_{n,n'}g^{(\beta)}_{ij,kl},$$
$$\overline{\left[\tilde{\Phi}_n^{(\pm)}\right]_{ij}\left[\tilde{\Phi}_{n'}^{(\mp)}\right]_{kl}} = 0, \quad (34)$$

which consists of independent random matrices from the appropriate GE. The representation (33) is advantageous when the number of $\tilde{f}_n > 0$ is finite since it can then be easily employed in a numerical construction of the GP, as we do in the following sections. Note that any process so constructed would necessarily have $\eta = 2$.

Finally, the corresponding x-space representation (17) in terms of the EGP $\Phi(x)$ is

$$H(x) = \int_0^L \frac{dx'}{L}\left[\sqrt{\tilde{f}_0} + 2\sum_{n>0}\sqrt{\tilde{f}_n}\cos\frac{2\pi n(x-x')}{L}\right] \times\Phi(x'). \quad (35)$$

C. The discrete Gaussian process

An actual construction of a process $H(x)$ on a computer requires a representation in terms of a finite number of random elements. An arbitrary correlation function $f(x)$ has in general infinitely many Fourier components $\tilde{f}_n > 0$ when restricted to a finite interval $x \in [0, L]$. To make their number finite we must use a discrete process $H_m \equiv H(x_m)$ with a finite number of points x_m. We take $2M+1$ equally spaced points $x_m = mL/(2M+1)$, $m = 0, \ldots, 2M$, and write

$$f(x_m - x_{m'}) = \frac{\tilde{f}_0}{2M+1} + \frac{2}{2M+1} \times\sum_{n=1}^{M}\tilde{f}_n\cos\frac{2\pi n(x_m - x_{m'})}{L}, \quad (36)$$

exploiting the discrete orthonormality of the functions $\exp\left(i\frac{2\pi n}{L}x_m\right)/\sqrt{2M+1}$ $(n = 0, \ldots, 2M)$ over the set of quadrature points x_m. Following the previous discussion, we obtain the finite construction for H_m

$$H_m = \sqrt{\frac{\tilde{f}_0}{2(2M+1)}}\tilde{\Phi}_0^{(+)} + \sum_{n=1}^{M}\sqrt{\frac{2\tilde{f}_n}{2M+1}}\left[\cos\left(\frac{2\pi nm}{2M+1}\right)\tilde{\Phi}_n^{(+)} + \sin\left(\frac{2\pi nm}{2M+1}\right)\tilde{\Phi}_n^{(-)}\right]. \quad (37)$$

Note that since we continue $f(x)$ from the interval $[-L/2, L/2]$, periodically only $H_m, H_{m'}$ with $|m-m'| \le M$ have the required correlation. Using (37) it is easy to generate a GP for any given $f(x)$ with $0 < \eta \le 2$ (see Sec. III).

Given a continuous process $H(x)$, any partial set of points x_m (not necessarily equally spaced) gives rise to a discrete GP $H_m \equiv H(x_m)$, since any marginal distribution of a Gaussian distribution remains Gaussian. To construct a discrete GP when x_m are not equally spaced we write

$$f_{mm'} \equiv f(x_m - x_{m'}) = \sum_{n=1}^{2M+1}\lambda_n P_{mn}P_{m'n}, \quad (38)$$

where $\lambda_n \ge 0$ are the eigenvalues of the matrix $f_{mm'}$ and P_{mn} is the orthogonal matrix that diagonalizes $f_{mm'}$. We then obtain the construction

$$H_m = \sum_{n=1}^{2M+1}P_{mn}\sqrt{\lambda_n}\bar{\Phi}_n \quad (39)$$

in analogy with (37), where $\bar{\Phi}_n$ are independent and belong to the appropriate GE. The probability distribution

$$P[H_m] \propto \exp\left(-\frac{\beta}{2a^2}\sum_{m,m'=1}^{2M+1}K_{mm'}\mathrm{Tr}H_mH_{m'}\right)$$

is obtained from

$$K_{mm'} = \sum_{n=1}^{2M+1}{}'\lambda_n^{-1}P_{mn}P_{m'n}.$$

In the case of the discrete Ornstein-Uhlenbeck GP (19), where $f_{mm'} = \exp(-\gamma|x_m - x_{m'}|)$, $P[H_m]$ can be cal-

culated analytically using the Markov property of the process to give

$$P[H_m] \propto \exp\left\{-\frac{\beta}{2a^2}\mathrm{Tr}\left[H_0^2+\sum_{m=2}^{2M+1}(H_m - f_{m,m-1}H_{m-1})^2/(1-f_{m,m-1}^2)\right]\right\}. \quad (40)$$

For equal spacings the eigenvalues λ_n reduce to the Fourier coefficients $\tilde{f}_n$ of (36) and the eigenvectors P_{mn} become linear combinations of $\exp\left(i\frac{2\pi n}{L}x_m\right)$.

D. The differentiable Gaussian process

In most physical applications the Hamiltonian is an analytic or at least a differentiable function of the parameter x. It is therefore important to identify the class of GPs that are differentiable. A differentiable GP is defined by the property that almost every one of its members $H(x)$ is a differentiable function of x with a continuous derivative dH/dx. We now show that this class consists of the processes with $\eta = 2$.

To see that any GP with $\eta = 2$ is differentiable, note first that in this case

$$\lim_{x\to 0}\frac{f(0)-f(x)}{x^2} < \infty . \quad (41)$$

Considering the quantity

$$\overline{\left|\frac{H_{ij}(x+\epsilon)-H_{ij}(x)}{\epsilon}-\frac{H_{ij}(x+\delta)-H_{ij}(x)}{\delta}\right|^2}, \quad (42)$$

it is easy to show that it approaches zero as ϵ and δ both approach zero independently, by using (1) to express the left-hand side of (42) in terms of the $f(x)$ and then using (41). It follows that for each realization of the GP, the quantity $[H(x+\epsilon)-H(x)]/\epsilon$ converges in the mean to a process, which we define to be the derivative process dH/dx,

$$\lim_{\epsilon\to 0}\overline{\left|\frac{H_{ij}(x+\epsilon)-H_{ij}(x)}{\epsilon}-\frac{dH_{ij}}{dx}\right|^2} = 0 . \quad (43)$$

Next, to see that dH/dx is continuous note that

$$\overline{\left|\frac{dH_{ij}}{dx}(x)-\frac{dH_{ij}}{dx}(x')\right|^2} \propto f''(0)-f''(x-x')$$

and approaches zero as $x' \to x$ if the second derivative $f''(x)$ is continuous at $x = 0$. The existence and continuity of $f''(x)$ for all x follows from $f''(x) = -\int e^{ikx}k^2\tilde{f}(k)dk$, whose absolute convergence is guaranteed by (41).

The converse is also true: if a GP is differentiable with $\overline{|\,dH_{ij}/dx\,|^2} < \infty$, then it is a $\eta = 2$ process. Indeed, in this case

$$\lim_{x\to 0}\frac{f(0)-f(x)}{x^2} = \frac{2\beta}{a^2}\frac{1}{g_{ijji}^{(\beta)}}\overline{\left|\frac{dH_{ij}}{dx}\right|^2} < \infty , \quad (44)$$

implying $\eta \geq 2$. However, a GP with $\eta > 2$ would have $\overline{|\,dH_{ij}/dx\,|^2} = 0$ and consequently $H_{ij}(x) = \text{const}$ for almost all members of the process, corresponding to $\eta = 0$ instead. Hence this GP must have $\eta = 2$. For example, although the derivative of the $\eta = 1$ Ornstein-Uhlenbeck GP (20) exists: $dH/dx = -\gamma H(x) + \sqrt{2\gamma}\Phi(x)$, it is not continuous and its variance $\overline{|\,dH_{ij}/dx\,|^2}$ diverges.

As a consequence of the nonexistence of the $\eta > 2$ processes, the Fourier transform $\tilde{f}(k)$ of any correlation function $f(x)$ whose short-distance behavior (28) has $\eta > 2$ is negative for some k and the corresponding GP measure is unnormalizable. This is confirmed numerically in Sec. III.

E. The conditional two-matrix distribution

We shall be interested in calculating two-point correlation functions of the form

$$c(x,x') = \overline{\tilde{\mathcal{O}}(x)\tilde{\mathcal{O}}(x')}, \qquad \tilde{\mathcal{O}}(x) = \frac{\mathcal{O}(x)-\overline{\mathcal{O}}}{\sqrt{\overline{\mathcal{O}^2}-\overline{\mathcal{O}}^2}}, \quad (45)$$

where $\mathcal{O}(x)$ is any function of the spectrum and eigenfunctions of $H(x)$ (i.e., its matrix elements). Since the process is translational invariant $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(x')} = \overline{\mathcal{O}}$. The correlator $c(x,x')$ depends only on $H(x), H(x')$ for $x \neq x'$ (for $x = x'$, $c = 1$) and is determined by the joint two-matrix distribution

$$P(H(x),H(x')) \propto \exp\left\{-\frac{\beta}{2a^2}\mathrm{Tr}\{[H(x)]^2+[H(x')]^2 - 2fH(x)H(x')\}/(1-f^2)\right\} \quad (46)$$

rather than by the full process distribution (4). Here $f \equiv f(|x-x'|)$, implying that $c(x,x') = c(|x-x'|)$. The distribution (46) can be obtained by integrating over all matrix elements of $H(x'')$ where $x'' \neq x, x'$ in (4). All integrations are Gaussian and can be performed exactly. However, in order to derive (46) it is sufficient to observe that these integrations result in a Gaussian distribution and therefore has the general form

$$P(H(x),H(x')) \propto \exp\left(-\mathrm{Tr}\{u_{11}[H(x)]^2+u_{22}[H(x')]^2 - 2u_{12}H(x)H(x')\}\right). \quad (47)$$

The coefficients in (47) are found by requiring that (1) be satisfied, or equivalently

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22}\end{pmatrix}^{-1} = \frac{2a^2}{\beta}\begin{pmatrix} 1 & f \\ f & 1\end{pmatrix}, \quad (48)$$

which gives (46). For actual calculations of GP averages involving $H(x)$ and $H(x')$, which are carried out below,

it is useful to also have the conditional two-matrix distribution

$$P(H(x') \mid H(x))$$

$$= P(H(x), H(x'))/P(H(x))$$
$$\propto \exp\left\{-\frac{\beta}{2a^2}\mathrm{Tr}\left[H(x') - fH(x)\right]^2/(1-f^2)\right\} . \quad (49)$$

III. SCALING AND UNIVERSALITY

Consider the parametric correlator (PC) $c(x-x')$ in (45). In general it depends on the parameters a, $f(x-x')$, and N, which characterize the GP. In this section we present a scaling procedure that eliminates this dependence and results in a GP-independent form of $c(x-x')$. We shall demonstrate that this form is universal. Although calculating $c(x-x')$ in a given physical system produces a system-dependent function, upon scaling it becomes system independent and identical to the scaled GP result. For $\eta = 2$, the scaling reduces to that introduced in [10].

To make a correspondence between the GP and a particular physical system we recall that a sets the mean level spacing Δ at the center of the spectrum through $\Delta = 1/\overline{\rho(0)} = \pi a/\sqrt{2N}$, where $\rho(E)$ is the level density. It is easy to show that the energy scaling (unfolding) $E_i \to \epsilon_i = E_i/\Delta$ leaves $c(x-x')$ independent of a. Indeed the joint two-matrix distribution (46) can be rewritten explicitly in terms of the unfolded energies ϵ_i, ϵ_i' and eigenfunctions ψ_i, ψ_i' of $H(x), H(x')$

$$P(H(x), H(x')) \propto \exp\left\{-\frac{\pi^2\beta}{4N}\left[\sum_i(\epsilon_i^2+\epsilon_i'^2) - 2f\sum_{i,j}\epsilon_i\epsilon_j'|\langle\psi_i \mid \psi_j'\rangle|^2\right]\Big/(1-f^2)\right\} . \quad (50)$$

Thus for any quantity $\mathcal{O}(\epsilon_i, \psi_i)$, the average $\overline{\mathcal{O}\mathcal{O}'}$ taken using (50) is a independent.

Next we wish to perform a scaling that eliminates the dependence on $f(x-x')$. We begin by writing down the expansion of f to leading order in $x-x'$

$$f(x-x') \approx 1 - \kappa|x-x'|^\eta , \quad (51)$$

with the constants κ and $\eta > 0$ whose significance will be discussed below. Following [10], which introduced the variance of the level velocity $\overline{(\partial\epsilon_i/\partial x)^2}$, we shall show more generally that the quantity $\overline{\Delta\epsilon_i^2}/\Delta x^\eta$ in the limit $\Delta x \to 0$ emerges as the relevant one to consider [16,24]. To calculate the average $\overline{\Delta E_i^2}$ we use first-order perturbation theory $\Delta E_i = \langle\psi_i|H' - H|\psi_i\rangle$, where ψ_i are the eigenfunctions of $H = H(x)$ with energies E_i and $H' = H(x')$. Motivated by the functional form of the conditional probability measure (49) we rewrite $H' - H = H' - fH - (1-f)H$ and obtain

$$\Delta E_i^2 = |\langle\psi_i|H' - fH|\psi_i\rangle|^2 + (1-f)^2E_i^2 - 2(1-f)E_i\,\mathrm{Re}\langle\psi_i|H'-fH|\psi_i\rangle . \quad (52)$$

We perform the GP averaging of (52) in two steps. First, using the conditional distribution (49) we average over H' while keeping H fixed. The first term on the right-hand side of (52) gives $a^2(1-f^2)/\beta$ and the last term vanishes; hence as $N \to \infty$

$$\overline{\Delta E_i^2} = \frac{2a^2\kappa}{\beta}|\Delta x|^\eta + O\left(|\Delta x|^{2\eta}\right) . \quad (53)$$

Equation (53) describes how the energy levels diffuse as the parameter varies. For the diffusive behavior of the scaled levels we get

$$\overline{\Delta\epsilon_i^2} = D|\Delta x|^\eta + O\left(|\Delta x|^{2\eta}\right) , \quad (54)$$

where we define

$$D = \lim_{\Delta x\to 0}\frac{\overline{\Delta\epsilon_i^2}}{\Delta x^\eta} = \frac{4N\kappa}{\pi^2\beta} . \quad (55)$$

D plays the role of the short-time diffusion constant. The case $\eta = 1$ is that of ordinary diffusion and corresponds to Dyson's Brownian motion model [17], which has $f(t-t') = \exp(-\gamma|t-t'|) \approx 1 - \gamma|t-t'|$. For $\eta \neq 1$ we have anomalous diffusion [18].

The level diffusion law (54) motivates the parameter scaling

$$x \to \bar{x} = D^{1/\eta}x . \quad (56)$$

We now show that the diffusion scaling (56) eliminates the dependence of $c(x-x')$ (45) on f. In terms of $\bar{x}$, the short-distance behavior of the process correlation function given in (51) is

$$f \approx 1 - \frac{\beta\pi^2}{4N}|\bar{x}-\bar{x}'|^\eta . \quad (57)$$

In Sec. IV we shall see that the correlations measured by $c(\bar{x})$ decay at $\bar{x} \sim 1$; therefore for large N the short-distance expansion of f in (51) and (57) is justified. In addition, we make the following *conjecture*: for large N, all PCs $c(\bar{x}-\bar{x}')$ [see (45)] depend on N and f only through the combination $N[1-f(\bar{x}-\bar{x}')]$. Using (51) and (57) we have

$$N(1-f) = N\kappa|x-x'|^\eta = \frac{\beta\pi^2}{4}|\bar{x}-\bar{x}'|^\eta ; \quad (58)$$

thus it follows from our conjecture that all PCs are universal functions of $|\bar{x}-\bar{x}'|^\eta$. This is the main result of this section.

Although we do not have a rigorous proof for the above conjecture, we have strong evidence in its favor. Two PCs that can be calculated analytically, the density correlator (64) and the energy-dependent parametric overlap (74), explicitly show $N(1-f)$ dependence. Numerical demonstrations of the universality of several PCs after

the scaling (56) also support this conjecture. These two forms of evidence are discussed in Sec. IV.

Here we give a physical "derivation" of this conjecture. We first concentrate on the Dyson process (23) and argue that its PCs are independent of N in the limit of large N if we choose the friction constant $\gamma \propto \frac{1}{N}$. According to Dyson [17] the eigenvalues of $H(t)$ describe a diffusive Coulomb gas that has two distinct time scales: the microscopic time scale $t \sim (\gamma N)^{-1}$ to reach local thermodynamic equilibrium and the macroscopic time scale $t \sim \gamma^{-1}$ that is required to reach global equilibrium. To obtain N-independent correlations, it is necessary that the *microscopic* time scale will be independent of N, which leads us to the choice $\gamma = \frac{\gamma_1}{N}$. A GP with a general correlation f would lead to correlations similar to the Dyson process if we can identify $f(x-x') \leftrightarrow \exp\left(-\frac{\gamma_1}{N}|t-t'|\right)$. In the limit of large N and upon using (51) this is equivalent to $N(1-f) = \frac{\beta\pi^2}{4}|\bar{x}-\bar{x}'|^\eta \leftrightarrow \gamma_1|t-t'|$. Thus the PCs are completely determined by the combination in (58).

For the special case of $\eta = 2$ in (51), the diffusion scaling factor $\sqrt{D}$ in (55) and (56) is just the level velocity scaling factor

$$D = C(0) = \overline{\left(\frac{\partial \epsilon_i}{\partial x}\right)^2} . \tag{59}$$

The $C(0)$ scaling was first discussed in [10], where it was derived using the supersymmetry method and shown to lead to the universality of two spectral PCs, the level velocity (numerically) and the level density (analytically). Beyond spectral correlations, two PCs involving the eigenfunctions that could be calculated analytically, the oscillator strength function PC [25] and the conductance PC in open multichannel quantum dots [26], were also shown to become universal upon the $C(0)$ scaling, all using the supersymmetry method [12]. The GP framework has the advantage that the universality of all PCs under this scaling emerges quite simply and naturally. Furthermore, in this framework it is evident that the level velocity scaling is not the correct procedure for $\eta \neq 2$. In particular, for $\eta < 2$ the process is not differentiable and the scaling factor of Ref. [10] $C(0) \equiv \overline{\Delta\epsilon^2/\Delta x^2}$ diverges as $C(0) \propto \Delta x^{\eta-2}$ [see (54)]. For $\eta > 2$ we conclude from (54) that $\overline{(\partial\epsilon_i/\partial x)^2} = 0$, in accord with our result in Sec. II D.

In order to demonstrate the use of the more general scaling (56) we construct numerically several discrete GPs using the construction method described in Sec. II C [see (36) and (37)] with the correlation function

$$f(x) = e^{-\kappa|x|^\eta} , \tag{60}$$

which has the short-distance behavior (51). We find numerically that the Fourier components $\tilde{f}_k$ are nonnegative only for $\eta \leq 2$ (see Sec. II D). Figure 1 shows the energy levels as a function of $\bar{x}$ for a typical member $H(x)$ of the GP for $\eta = 1, 3/2$, and 2. For $\eta = 2$ the energy levels are smooth and differentiable as a function of x. However, as η decreases the energy levels show a more irregular behavior with stronger fluctuations at small scales. This apparent indifferentiability is consistent with the divergence of $C(0)$ for $\eta < 2$. Nevertheless, from the above discussion it follows that the level diffusion PC $\overline{[\epsilon_i(x) - \epsilon_i(x')]^2}$ is a universal function of $|\bar{x}-\bar{x}'|^\eta$. Its functional form at short distances is obtained by the scaling of (54)

$$\overline{\Delta\epsilon_i^2} = |\Delta\bar{x}|^\eta + O\left(|\Delta\bar{x}|^{2\eta}\right) \tag{61}$$

and manifests its independence of any parameter but time-reversal symmetry status. We verify this universality by computing $\overline{\Delta\epsilon_i^2}$ at different N, κ, and η. The results are presented in Fig. 2 and show a universal form in the different cases when plotted as a function of $|\Delta\bar{x}|^\eta$. We also demonstrate that the short-distance behavior is

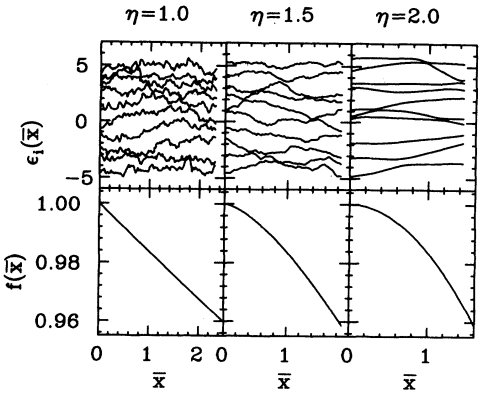

FIG. 1. Top: several energy levels from the middle of the spectrum as a function of $\bar{x}$ for different values of η, obtained from a simulation of the GP (37) with the process correlation function $f(x) = e^{-|x|^\eta}$ (60) for $\eta = 1, 3/2$ and the GP (63) for $\eta = 2$. The values $N = 150$ and $\kappa = 0.5$ are used. Bottom: the function $f(x)$ (60).

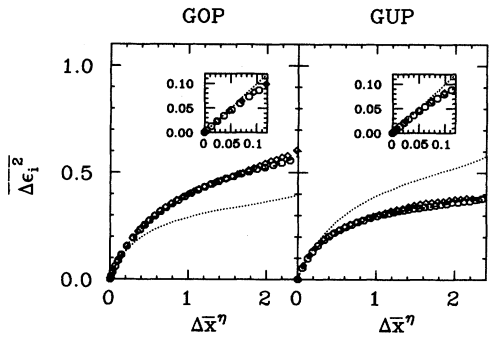

FIG. 2. Level-diffusion correlator $\overline{[\epsilon_i(x) - \epsilon_i(x')]^2}$ as a function of $|\bar{x} - \bar{x}'|^\eta$ in the orthogonal (left) and unitary (right) cases, obtained from 100 simulations of the GP (37) with the process correlation function (60). The values $N = 150$, $\kappa = 0.5$, and $\eta = 1$ (diamonds) and $N = 100$, $\kappa = 1.0$, and $\eta = 1.5$ (circles) were used, taking the middle third of the spectrum in each case. The dashed line is calculated from the simple $\eta = 2$ GP (63). The dotted line shows for each symmetry class (i.e., orthogonal or unitary) the universal curve of the other one. The inset shows the short-distance behavior (61) predicted to be linear with a unit slope (dotted).

linear with a unit slope, a result that is essential to the scaling (56) used throughout this paper.

Finally, we would like to point out the N scaling in the GP. Comparing the calculations of a PC within two GPs that are identical except for their values of N, it follows from (55) that the scaling $x \to \bar{x} = N^{1/\eta}x$ would make them equivalent. This will also be shown explicitly for a particular PC (with $\eta = 2$) in the next section and means that the typical correlation length in the GP scales like $N^{-1/\eta}$.

IV. UNIVERSAL CORRELATORS

In this section we calculate several correlators and demonstrate their universality upon the scaling (56) within the GP framework. We reproduce the spectral correlators of [10], but more importantly we obtain new parametric correlators that involve the eigenfunctions for both the orthogonal [16] and unitary cases. Some of these correlators are computed from numerical simulations of a simple GP and others are calculated analytically using the supersymmetry method of Ref. [12].

A. A simple GP and spectral correlations

The level velocity PC

$$c(x-x') = \overline{\tilde{v}(x)\tilde{v}(x')}\ ,$$
$$\tilde{v}(x) = \frac{v(x)-\overline{v}}{\sqrt{\overline{v^2}-\overline{v}^2}}\ , \qquad v(x) = \frac{\partial \epsilon_i}{\partial x}(x) \tag{62}$$

was introduced in [10], where it was shown to become universal after the parameter is scaled by the rms level velocity $\sqrt{C(0)} = \sqrt{D}$ [see (59)]. This result was based on numerical simulations of the Anderson model and the universal form of the level velocity correlator $c(\bar{x})$ was obtained in the same manner. However, in the GP framework, our argument in Sec. III for the universality of all correlators implies the universality of $c(x)$ as well and its universal form can be computed directly from any GP. Here and in all the following computations of universal PCs, we use the simple $\eta = 2$ GP defined by [27]

$$H(x) = H_1 \cos x + H_2 \sin x\ , \tag{63}$$

where H_1, H_2 are independent $N \times N$ random matrices drawn from the appropriate GE. Equation (63) is a special case of the construction (33) with $L = 2\pi$ and $f(x) = \cos x$. For these computations we generate 300 GP simulations with $N = 150$, using only the middle third of the states where the level density is approximately flat. The average is taken over the different states, parameter values, and simulations.

To illustrate the possibility of computing any universal PC using the GP (63) we first use it to obtain the known universal form of the velocity PC $c(\bar{x})$. Figure 3 shows the results for both the orthogonal and the unitary cases, which are identical to those of Ref. [10]. As the parameter varies, the velocities become anticorrelated due to the oscillations of the levels and later decorrelate. The decorrelation is accelerated by the breaking of time-reversal symmetry.

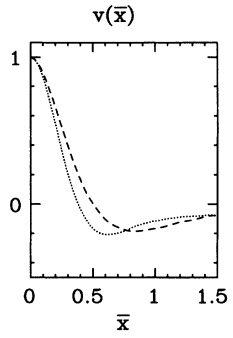

FIG. 3. Universal form of the level velocity correlator $c(x-x')$ (62) for the orthogonal (dashed) and unitary (dotted) cases, obtained from 300 simulations the GP (63) with $N = 150$ using the middle third of the spectrum. The results are identical to those of Ref. [10].

The level density PC

$$k(E-E', x-x') = \overline{\tilde{\rho}(E,x)\tilde{\rho}(E',x')}\ ,$$
$$\tilde{\rho}(E,x) = \Delta\left[\rho(E,x) - \overline{\rho}\right] = \Delta\rho(E,x) - 1 \tag{64}$$

was derived analytically in Ref. [10] for an electron diffusing in a random potential in the presence and absence of a magnetic field. This PC was also calculated in [13] for the random-matrix model $H(x) = H_0 + xU$, where H_0 is a random matrix from the appropriate GE and U is a fixed traceless matrix, giving the same universal forms.

Now we calculate the density PC for the general GP (1). Our purpose in repeating the calculation for this more general case is to show that the conjecture given in Sec. III indeed holds in this case, namely, that the N dependence of this PC is through $N(1-f)$, which implies that the scaling (56) results in a universal function of $|\bar{x}-\bar{x}'|^\eta$ that is independent of N. The density PC (64) can be written as

$$k(E-E', x-x') = \frac{\Delta^2}{2\pi^2}\mathrm{Re}\left[\overline{\mathrm{Tr}G(E^-,x)\mathrm{Tr}G(E'^+,x')} - \overline{\mathrm{Tr}G(E^-,x)\mathrm{Tr}G(E'^-,x')}\right] - 1\ , \tag{65}$$

where $G(E^\pm,x) = [E \pm i\delta - H(x)]^{-1}$ are the advanced and retarded Green functions. Using the supersymmetry method [12,28] to calculate averages of product of Green functions, we find for the GOP case

$$k^{(\beta=1)}(E-E', x-x') = \mathrm{Re}\int_{-1}^{1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \frac{(1-\lambda^2)(\lambda_1\lambda_2-\lambda)^2}{(\lambda_1^2+\lambda_2^2+\lambda^2-2\lambda_1\lambda_2\lambda-1)^2} \exp[i\pi(\omega+2i\delta)(\lambda_1\lambda_2-\lambda) - N\kappa|x-x'|^\eta(2\lambda_1^2\lambda_2^2-\lambda_1^2-\lambda_2^2-\lambda^2+1)]\ , \tag{66}$$

where $\omega = (E-E')/\Delta$. The expression for the GUP case is simpler

$$k^{(\beta=2)}(E-E', x-x')$$

$$= \frac{1}{2}\mathrm{Re}\int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_1 \exp[i\pi(\omega + 2i\delta)(\lambda_1 - \lambda) - N\kappa|x-x'|^{\eta}(\lambda_1^2 - \lambda^2)] \,. \quad (67)$$

We note that the density PC in both cases is independent of a and depends on N only through $N(1-f) = N\kappa \mid x-x' \mid^{\eta}$ [see (58)], in accord with our conjecture. In particular, for $\eta = 2$ the substitution $N\kappa|x-x'|^{\eta} = (\beta\pi^2/4)(\bar{x}-\bar{x}')^2$ in Eqs. (66) and (67) gives the universal forms calculated in [10,13].

B. Parametric correlations of eigenfunctions

In Sec. III we argued the universality of all PCs, including those involving the eigenfunctions. We proceed to demonstrate this point by studying several eigenfunction correlators both numerically and analytically.

The first is the averaged parametric overlap

$$o(x-x') = \overline{|\langle\psi_i(x)|\psi_i(x')\rangle|^2} \,, \quad (68)$$

which measures the decrease in the squared overlap of an eigenfunction at x and the same eigenfunction at x' from $o(0) = 1$ as x' separates from x. The second is the projection PC

$$p(x-x') = \overline{\langle\phi|\psi_i(x)\rangle^* \langle\phi|\psi_i(x')\rangle} \,/\, \overline{|\langle\phi|\psi_i\rangle|^2} \,, \quad (69)$$

which measures the correlation between the projections of ψ_i on a normalized fixed state ϕ at different values of x. Note that $\langle\phi|\psi_i\rangle$ averages to zero so that (69) is of the general form (45). The projection PC is independent of the choice of ϕ. Indeed, if we chose another vector ϕ' then there would have existed a unitary transformation U satisfying $|\phi'\rangle = U|\phi\rangle$, which could have been used to rotate the Hamiltonian matrix $H(x)$ at each x into $H'(x) = UH(x)U^{\dagger}$ with eigenfunctions $|\psi_i'(x)\rangle = U|\psi_i(x)\rangle$. Since the GP probability measure (4) is invariant under a global rotation, we have $p_{\phi}(x-x') = p_{\phi'}(x-x')$. In particular, the fixed state can be taken to be a position eigenstate $|\phi\rangle = |\vec{r}\rangle$, in which case $p(x-x')$ measures the correlation between eigenfunctions belonging to different values of x at a given space point $\vec{r}$:

$$p(x-x') = \overline{\psi_i^x(\vec{r})^* \psi_i^{x'}(\vec{r})} \,/\, \overline{|\psi_i(\vec{r})|^2} \,. \quad (70)$$

We note that the projection PC $p(x-x')$, unlike the parametric overlap $o(x-x')$, does depend on the choice of the eigenfunction phases. For the orthogonal case, however, $\psi_i(x)$ are real and determined up to a sign that is kept fixed as x varies: Since

$$\langle\psi_i(x)|\psi_i(x')\rangle = 1 - O((x-x')^2) \,, \quad (71)$$

the sign of $\psi_i(x')$ is determined so that $\langle\psi_i(x)|\psi_i(x')\rangle > 0$, while $x'-x$ is made sufficiently small, to ensure that there is no avoided level crossing in the interval (x, x').

We used the GP (63) to compute the universal forms of $o(x-x')$ and $p(x-x')$. The results are given in the upper panels of Fig. 4. We find that these correlators are approximated very well by simple functions. The parametric overlap is given by

$$o(\bar{x}-\bar{x}') \approx \left[\frac{1}{1+(\bar{x}-\bar{x}')^2/a_{\beta}^2}\right]^{\beta} \,, \quad (72)$$

namely, a Lorentzian of width $a_1 = 0.48 \pm 0.03$ in the orthogonal case and a squared Lorentzian of width $a_2 = 0.64 \pm 0.04$ in the unitary case, with a χ^2 per degree of freedom of approximately 10^{-2}. Thus the eigenfunctions decorrelate faster in the unitary case than in the orthogonal case in qualitative agreement with the level velocity correlator. The projection correlator in the orthogonal case is given by

$$p(\bar{x}-\bar{x}') \approx \left[\frac{1}{1+(\bar{x}-\bar{x}')^2/b^2}\right]^{2} \,, \quad (73)$$

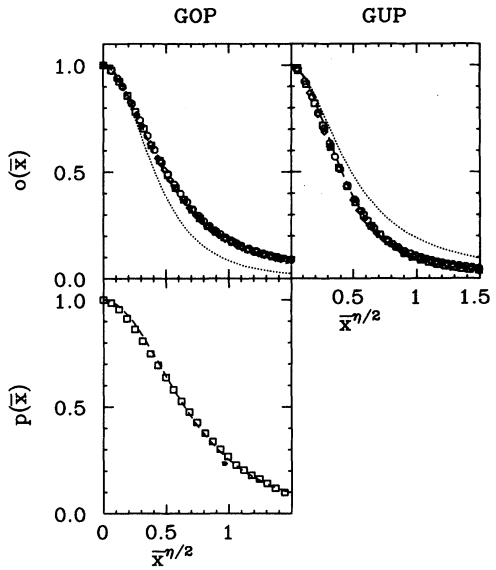

FIG. 4. Universal forms of the parametric overlap $o(x-x')$ (68) (top) and the projection correlator $p(x-x')$ (69) (bottom) for the orthogonal (left) and unitary (right) cases. The squares are the results obtained from simulations of the $\eta = 2$ GP (63). The dashed lines show the approximations (72) and (73). The dotted line shows for each symmetry class (i.e., orthogonal or unitary) the universal curve of the other one. Also shown is the parametric overlap $o(x-x')$ as a function of $|\bar{x}-\bar{x}'|^{\eta/2}$ obtained from 100 simulations of the GP (37) with the process correlation function (60). The values $N = 150$, $\kappa = 0.5$, and $\eta = 1$ (diamonds) and $N = 100$, $\kappa = 1.0$, and $\eta = 1.5$ (circles) were used, taking the middle third of the spectrum in each case.

with $b = 1.00 \pm 0.08$. a_1, a_2, b were obtained by least-squares fits.

It is interesting to verify that these universal forms are valid also for processes with $\eta \neq 2$, as expected from our general discussion in Sec. III. For that purpose we used the GP (37) with the correlation (60) to compute the parametric overlap $o(x-x')$ as a function of $|\bar{x}-\bar{x}'|^{\eta/2}$ for different values of N, κ, and η in the orthogonal and unitary cases. The results are presented in Fig. 4 and show a complete agreement with the form (72) upon the replacement $|\bar{x}-\bar{x}'| \to |\bar{x}-\bar{x}'|^{\eta/2}$.

Finally, we introduce a third PC, which we calculate analytically. This PC, the energy-dependent parametric overlap $o(E-E', x-x')$, is related to the parametric overlap $o(x-x')$ (68) but involves both the energies and the eigenfunctions. It is given by

$$o(E-E', x-x') = \Delta^2 \overline{\sum_{i,j} |\langle \psi_i(x)|\psi_j(x')\rangle|^2 \delta(E-E_i(x))\delta(E'-E_j(x'))} \tag{74}$$

and measures the averaged parametric overlap of eigenfunctions whose corresponding energies are separated by $E-E'$. Similarly to the density PC (64), we calculate $o(E-E', x-x')$ using the supersymmetry method [12] and therefore start by expressing it in terms of Green functions

$$o(E-E', x-x') = \frac{\Delta^2}{2\pi^2} \mathrm{ReTr}\left[\overline{G(E^-,x)G(E'^+,x')} - \overline{G(E^-,x)G(E'^-,x')}\right] . \tag{75}$$

This calculation is described in the Appendix. For the GOP case, we find

$$o^{(\beta=1)}(E-E', x-x') = \frac{1}{2}\mathrm{Re}\int_{-1}^{1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \frac{(1-\lambda^2)(2\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1)}{(\lambda_1^2+\lambda_2^2+\lambda^2-2\lambda_1\lambda_2\lambda - 1)^2} \times \exp\left[i\pi(\omega+2i\delta)(\lambda_1\lambda_2-\lambda) - N\kappa|x-x'|^\eta(2\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1)\right] , \tag{76}$$

where $\omega = (E-E')/\Delta$. The GUP expression is simpler:

$$o^{(\beta=2)}(E-E', x-x') = \frac{1}{4}\mathrm{Re}\int_{-1}^{1} d\lambda \int_1^\infty d\lambda_1 \frac{\lambda_1+\lambda}{\lambda_1-\lambda} \exp[i\pi(\omega+2i\delta)(\lambda_1-\lambda) - N\kappa|x-x'|^\eta(\lambda_1^2-\lambda^2)] . \tag{77}$$

As for the density PC (66) and (67), the final results here depend only on $N(1-f) = N\kappa|x-x'|^\eta$, confirming once again our conjecture and exhibiting a universal functional dependence on $|\bar{x}-\bar{x}'|^\eta$ upon the scaling (56).

The parametric overlap $o(E-E', x-x')$ clearly diverges at $E=E'$, $x=x'$ as $\delta \to 0$, as is also the case for the density PC (64). It is therefore convenient to define the normalized energy-dependent parametric overlap

$$\tilde{o}(\Omega, x-x') = \frac{\overline{\sum_{i,j} |\langle \psi_i(x)|\psi_j(x')\rangle|^2 \delta(E_i(x)+\Omega-E_j(x'))}}{\overline{\sum_{i,j} \delta(E_i(x)+\Omega-E_j(x'))}} = \frac{o(\Omega, x-x')}{k(\Omega, x-x')+1} , \tag{78}$$

with $\Omega = E-E'$. $\tilde{o}(\Omega, x-x')$ measures the weighted average of the parametric overlap $|\langle \psi_i(x)|\psi_j(x')\rangle|^2$ with a weight function $\delta(E_i(x)+\Omega-E_j(x'))$ that favors states separated by Ω. The universal forms of (78) are shown in Fig. 5 for $\delta = 0.05\Delta$ by the solid (GOP) and dotted (GUP) lines where we have used the analytic expressions (76) and (77). This correlator exhibits a peak that shifts to the right and diminishes as $\omega = \Omega/\Delta$ increases. This feature is expected since at small Ω the dominant term in the sums in both the nominator and the denominator

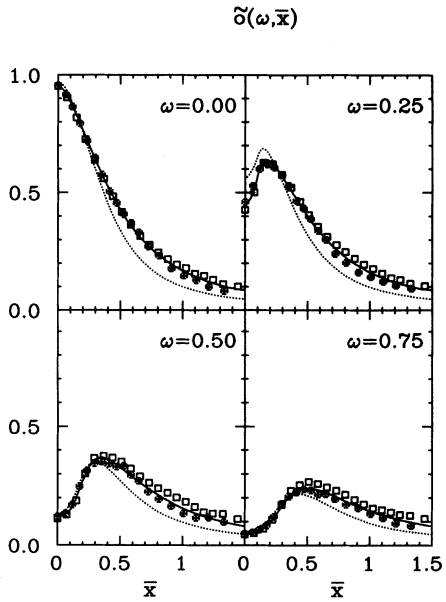

FIG. 5. Universal form of the normalized energy-dependent parametric overlap $\tilde{o}(\Omega, x-x')$ (78) for the Anderson model with time-reversal symmetry, compared with the GOP (solid) and GUP (dotted) predictions given by the analytical expressions in (76) and (77), respectively. The latter also agree with simulations of the GP (63) with $N = 150$. Results for the Anderson model are shown for a cylindrical geometry with $W = 1$ (squares) and a toroidal geometry with $W = 2$ (pluses). We take $\delta = 0.05\Delta$.

of (78) is $i = j$; thus for $E_j(x')$ to remain correlated $E_i(x) + \Omega$ as Ω increases, x' has to move further away from x. Note that correlations in the unitary case decay faster. Calculations based on simulations of the GP (63) give the same universal forms.

In addition to parametric correlators such as $o(x - x')$ we also expect the full probability density of the quantity of interest to become universal upon the same scaling. In particular, the overlap density

$$p_{x-x'}\left[|\langle\psi_i(x)|\psi_i(x')\rangle|^2 = y\right] , \tag{79}$$

of which $o(x - x')$ is the first moment, should become a universal function of y for a given value of $\bar{x} - \bar{x}'$. This will be demonstrated in the context of a physical model in Sec. V.

V. APPLICATION TO DISORDERED SYSTEMS: THE ANDERSON MODEL

In the remaining sections we demonstrate the applicability of the Gaussian random-matrix process to physical systems. We study both a disordered system (Anderson model) and a chaotic system. We compute the PCs introduced in Sec. IV in those systems, perform the scaling procedure, and compare them to the corresponding universal theoretical predictions of the GP.

The Anderson model [29] is a discretized version of Schrödinger's equation $\frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2\Psi + V\Psi = E\Psi$ for a charged particle moving in a potential $V(\vec{r})$ that has a random component, under the influence of a magnetic field. The discretization in two dimensions on a $n_x \times n_y$ square lattice with spacing s between neighboring sites leads to the matrix eigenvalue equations

$$\frac{2ms^2}{\hbar^2}V_{i,j}\Psi_{i,j} - e^{-i\theta^x_{i,j}}\Psi_{i+1,j}$$

$$-e^{i\theta^x_{i,j}}\Psi_{i-1,j} - e^{-i\theta^y_{i,j}}\Psi_{i,j+1} - e^{i\theta^y_{i,j}}\Psi_{i,j-1}$$

$$= \left[\frac{2ms^2}{\hbar^2}E - 4 - \left(\frac{qs}{\hbar c}\right)^2 \vec{A}^2_{i,j}\right]\Psi_{i,j} , \tag{80}$$

where $\vec{\theta}_{i,j} = \frac{qs}{\hbar c}\vec{A}_{i,j}$ and $i = 1, ..., n_x$, $j = 1, ..., n_y$. We fold the lattice into a cylinder whose symmetry axis is along x by imposing periodic boundary conditions

$$\Psi_{i,0} = \Psi_{i,n_y} , \qquad \Psi_{i,n_y+1} = \Psi_{i,1} . \tag{81}$$

If the magnetic field is pointed along the symmetry axis and is associated with a flux ϕ through the cylinder, the vector potential would have a magnitude of $A = \phi/n_y s$ and be directed around the cylinder, so $\theta^x_{i,j} = 0$ and $\theta^y_{i,j} = \frac{2\pi}{n_y}\phi/\phi_0 \equiv \theta$, where $\phi_0 = hc/q$ is the flux unit. Separating the potential into a fixed part u and a random part w: $\frac{2ms^2}{\hbar^2}V(\vec{r}) = u(\vec{r}) + w(\vec{r})$, we rewrite Eq. (80) as

$$(u_{i,j} + w_{i,j})\Psi_{i,j} - \Psi_{i+1,j} - \Psi_{i-1,j}$$

$$- e^{-i\theta}\Psi_{i,j+1} - e^{i\theta}\Psi_{i,j-1} = \epsilon\Psi_{i,j} . \tag{82}$$

The model is sometimes presented in a second quantized form

$$H = \sum_\alpha (u_\alpha + w_\alpha)a^\dagger_\alpha a_\alpha - \sum_{\langle\alpha\beta\rangle} e^{i\theta_{\alpha\beta}}a^\dagger_\alpha a_\beta , \tag{83}$$

with indices $\alpha = (i, j)$. The random on-site energies w_α are chosen uniformly from interval $[-W/2, W/2]$. The parameter W determines the degree of disorder in the system. For small W, the two-dimensional system is in the diffusive regime and the eigenfunctions are extended, whereas for a sufficiently large W the system crosses over to the localized regime [30].

We study the Anderson model using a 27×27 lattice in both a cylindrical and a toroidal geometry and different values of W to verify that the correlators we compute are insensitive to boundary conditions and to the degree of disorder, as long as it is sufficiently weak. In the orthogonal case we take a cylinder with $W = 1$ and a torus with $W = 2$. The parametric dependence is introduced by adding an external potential $u(\vec{r})$ in the form of a step

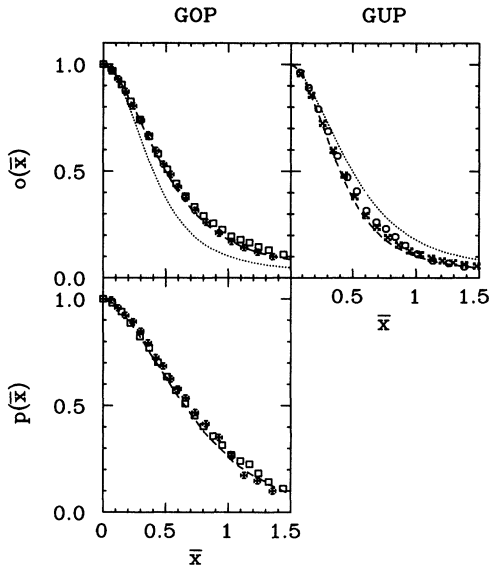

FIG. 6. Universal forms of the parametric overlap $o(x - x')$ (68) (top) and the projection correlator $p(x - x')$ (69) (bottom) in the Anderson model compared with the GP predictions (72) and (73) (dashed lines). Left: with time-reversal symmetry. Results are shown for a cylindrical geometry with $W = 1$ (squares) and a toroidal geometry with $W = 2$ (pluses), using the middle 200 eigenfunctions of the Hamiltonian (83) on a 27×27 lattice. Right: without time reversal symmetry. Results are shown for a cylindrical geometry with $W = 1$ and $x = u_0$ the strength of an external potential (circles) and a toroidal geometry with $W = 2$ and $x = B'$ the magnitude of an external field (crosses). For each symmetry class we show by dotted lines (top) the universal curve of the other one.

function along x [10]: $u(x,y) = u_0\Theta(x)$, with the potential strength u_0 serving as the parameter x. In Fig. 6 (left panels) we present the Anderson model simulations for $o(x-x')$ (68) and $p(x-x')$ (69) and compare them with the GOP prediction, reproduced by the simple functional forms (72) and (73), respectively. The agreement is very good for both geometries and degrees of disorder.

The unitary case is obtained by applying a magnetic field B associated with a flux $\phi = \phi_0/4$ [see the paragraph following (81)] along the cylinder axis. A parametric dependence is introduced either by an external potential as above or by folding the cylinder into a torus, applying an additional magnetic field B' perpendicular to the torus plane, and taking its magnitude as the parameter x. The latter situation amounts to taking $\theta^x_{i,j} = \left(n_x s^2/2\phi_0\right) B'$ in (80). Figure 6 (right) shows that these two situations produce curves for the scaled correlator $o(\bar{x}-\bar{x}')$, which are very close to each other and agree well with the GUP prediction.

We also compute the normalized energy-dependent parametric overlap $\tilde{o}(\Omega, x-x')$ (78) in the Anderson model for the orthogonal and unitary cases described above and compare the results with the corresponding GOP and GUP predictions given analytically by (76) and (77), respectively. The agreement is excellent for all cases, as shown in Figs. 5 and 7 for the orthogonal and unitary symmetries, respectively.

Finally, we compute the probability density $p_{x-x'}(y)$ of the overlap $y = |\langle\psi_i(x)|\psi_i(x')\rangle|^2$ [see Eq. (79)] as a function of y for several values of $\bar{x}-\bar{x}'$ in the Anderson model. In Figs. 8 and 9 we present the results in a histogram form for the orthogonal and unitary cases, respectively, and compare them with the GP predictions. At $\bar{x}-\bar{x}'=0$ we of course have $p(y)=\delta(y-1)$, whereas as the distance increases the distribution broadens and the peak shifts from $y=1$ to $y=0$ to manifest the

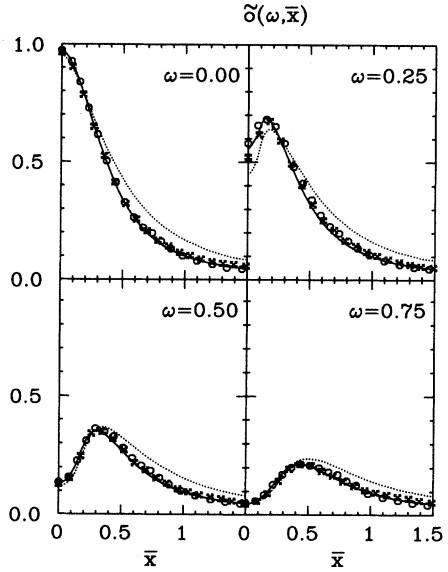

FIG. 7. Universal form of the normalized energy-dependent parametric overlap $\tilde{o}(\Omega, x-x')$ (78) for the Anderson model with broken time-reversal symmetry, compared with the GUP (dotted) prediction (77) (solid lines). Results are shown for a cylindrical geometry with $W=1$ and parametric dependence $x=u_0$ (circles) and a toroidal geometry with $W=2$ and $x=B'$ (crosses). We take $\delta = 0.05\Delta$. The dotted lines show corresponding curves (76) for the GOP.

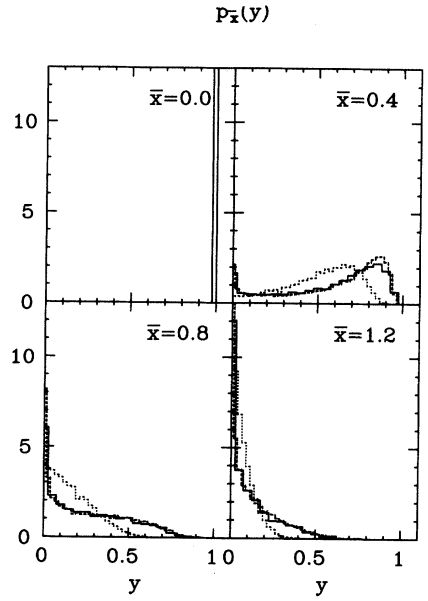

FIG. 8. Overlap probability density $p_{x-x'}$ $(|\langle\psi_i(x)|\psi_i(x')\rangle|^2 = y)$ (79) as a function of y at several values of $\bar{x}-\bar{x}'$ for the Anderson model with time-reversal symmetry (solid) compared with the GOP (dashed) and GUP (dotted) predictions. Results are shown for a cylindrical geometry with $W=1$.

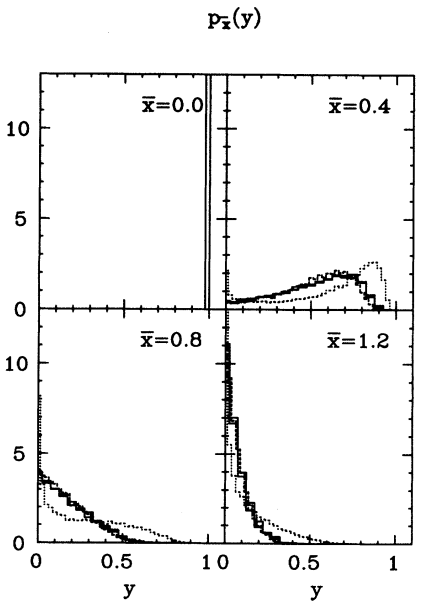

FIG. 9. Overlap probability density $p_{x-x'}$ $(|\langle\psi_i(x)|\psi_i(x')\rangle|^2 = y)$ (79) as a function of y at several values of $\bar{x}-\bar{x}'$ for the Anderson model with broken time-reversal symmetry (solid) compared with the GUP (dashed) and GOP (dotted) predictions. Results are shown for a cylindrical geometry with $W=1$ and parametric dependence $x=u_0$ and a toroidal geometry with $W=2$ and $x=B'$.

eigenstate decorrelation. Again we observe very good agreement with the theoretical GP predictions.

VI. APPLICATION TO CHAOTIC SYSTEMS: THE INTERACTING BOSON MODEL

In this section we demonstrate the success of the GP framework in describing universal parametric eigenfunction correlators in chaotic systems. For that purpose we use a nuclear structure model, the interacting boson model (IBM) [31]. Since this model has both regular and chaotic regimes, we can also study deviations from the universal correlators in the regular case.

The IBM is a phenomenological many-body model, which has been successful in describing spectra and electromagnetic transitions of low-lying collective states of medium- and heavy-mass even-even nuclei. In its simplest version, the IBM Hamiltonian is a quadratic function of the 36 generators of the u(6) Lie algebra $\{s^\dagger s, s^\dagger d_\mu, d^\dagger_\mu s, d^\dagger_\mu d_\nu; \mu,\nu = -2,...,2\}$. $s^\dagger, s$ and $d^\dagger_\mu, d_\mu$ are bosonic creation and annihilation operators that model nucleon pairs coupled to angular momenta of 0 and 2, respectively.

A compact parametrization of the IBM Hamiltonian is given in the framework of the self-consistent Q formalism [32]

$$H = c_0 n_d + c_1 \vec{L}\cdot\vec{L} + c_2 Q^\chi Q^\chi \,. \tag{84}$$

n_d, $\vec{L}$, and Q^χ are, respectively, the d-boson number, angular momentum, and quadrupole operators

$$\begin{aligned} n_d &= d^\dagger \tilde{d}\,, \\ \vec{L} &= \sqrt{10}(d^\dagger \times \tilde{d})^{(1)}\,, \\ Q^\chi &= (d^\dagger \times s + s^\dagger \times \tilde{d})^{(2)} + \chi (d^\dagger \times \tilde{d})^{(2)}\,, \end{aligned} \tag{85}$$

where $\tilde{d}_\mu = (-1)^\mu d_{-\mu}$.

The Hamiltonian (84) is known to be completely integrable [33] when it has a dynamical symmetry, i.e., when H can be written as a sum of Casimir invariant operators of a subalgebra chain

$$\begin{aligned} H &= a_0 C(g_0) + a_1 C(g_1) + a_2 C(g_2) + \cdots\,, \\ & g_0 \supset g_1 \supset g_2 \supset \cdots\,. \end{aligned} \tag{86}$$

In such a case the set of Casimir operators $C(g_i)$, complemented by invariant operators (when missing labels occur in the chain), form a complete set of mutually commuting constants of the motion, rendering the Hamiltonian completely integrable.

In the IBM we have g_0 =u(6), and since the nuclear Hamiltonian is invariant under rotations, each subalgebra chain should terminate with o(3). This restriction is fulfilled by three chains only

$$\mathrm{u}(6) \supset \left\{ \begin{array}{c} \mathrm{u}(5) \supset \mathrm{o}(5) \\ \mathrm{su}(3) \\ \mathrm{o}(6) \supset \mathrm{o}(5) \end{array} \right\} \supset \mathrm{o}(3)\,. \tag{87}$$

Chain I [u(5)] is obtained for $c_2 = 0$ and describes a spectrum of a vibrational nucleus. Chain II [su(3)] corresponds to $c_0 = 0, \chi = -\sqrt{7}/2$ and has a rotational spectrum. Chain III [o(6)] corresponds to $c_0 = 0, \chi = 0$ and describes a spectrum of a γ-unstable nucleus.

The IBM Hamiltonian H in (84) has a classical counterpart

$$\mathcal{H}(\vec{\alpha},\vec{\alpha}^*) = \langle\vec{\alpha}|H|\vec{\alpha}\rangle\,, \tag{88}$$

where

$$|\vec{\alpha}\rangle = e^{-\vec{\alpha}^2/2} \exp\left(\alpha_s s^\dagger + \sum_{\mu=-2}^{2} \alpha_\mu d^\dagger_\mu\right)|0\rangle$$

$$\vec{\alpha} = (\alpha_s, \alpha_{-2}, \ldots, \alpha_2) \tag{89}$$

are the Glauber coherent states for the s, d bosons and $|0\rangle$ is the vacuum state [34] and α_μ can be interpreted as the nuclear quadrupole shape variables. Canonical Hamilton's equations derived from (88), where $\vec{\alpha}$ and $i\vec{\alpha}^*$ play the role of conjugate coordinates and momenta, are equivalent to the time-dependent mean-field equations in the limit of large boson number $\mathcal{N}$. Therefore $1/\mathcal{N}$ plays the role of $\hbar$ in the quantum-classical correspondence for this model.

The chaotic dynamics of the IBM has been studied in [33] both classically and quantum mechanically. Here we study its parametric correlations in the regime $c_0 = 0$, $c_1 = 0$, $-0.8 \le \chi \le -0.5$, which lies in between

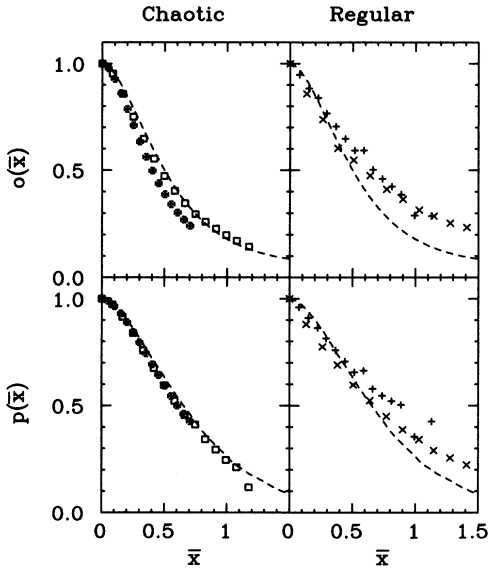

FIG. 10. Parametric overlap $o(x-x')$ (68) (top) and the projection correlator $p(x-x')$ (69) (bottom) in the IBM in a chaotic regime (left) and in a regular regime (right), compared with the GOP (dashed) predictions (72) and (73). In the chaotic case results are shown for $J = 2$ (pluses), where 80 states out of 117 are used, and $J = 6$ (squares), using 130 states out of 184. In the regular regime results are shown for $J = 2$ (crosses), where 80 states out of 117 are used, and for $J = 6$ (pluses) with 130 states out of 184.

the rotational ($\chi = -\sqrt{7}/2$) and γ-unstable ($\chi = 0$) dynamical symmetry limits. In this region the classical dynamics was found to be strongly chaotic and spectral fluctuations as well as electromagnetic $E2$ transition intensities distributions were in good agreement with RMT [33]. We compute $o(x-x')$ (68) and $p(x-x')$ (69) in the IBM for 25 bosons using χ as the parameter x. Since the total angular momentum J of a nucleus is a conserved quantum number, we can study these correlators for different values of J. The results for $J = 2$ and $J = 6$ are presented in Fig. 10 (top panels), where they are seen to be in good agreement with the GOP predictions (72) and (73).

In the IBM we can examine PCs in regular regimes as well, where deviations from the universal forms are expected. We considered the parameter regime between the γ-unstable and vibrational dynamical symmetry limits where $\chi = 0$ and replacing the parameters c_0, c_2 in (84) by $c_0 = \xi$, $c_2 = 1-\xi$ with $0 \le \xi \le 1$. In this regime there is a common o(5) symmetry and the Hamiltonian is completely integrable [33]. We computed $o(x-x')$ and $p(x-x')$ for $0.1 \le \xi \le 0.3$. The results after scaling are presented in Fig. 10 (bottom panels) and show a marked deviation from the GOP behavior. For small $\bar{x}$ the IBM curves are slightly below those of the GOP. For larger values of $\bar{x}$, however, the IBM correlators decay considerably more slowly than their universal counterparts, indicating the existence of stronger correlations in the regular case. Although the regular results are not expected to be universal, it would be interesting to study PCs in other regular systems and determine whether the qualitative features found here, in particular the slower decay of correlations, would turn out to be general.

VII. CONCLUSION

We have introduced the Gaussian random-matrix process, a direct generalization of Dyson's Gaussian ensembles, and shown that this framework is suitable for the discussion of parametric correlation functions. All GPs can be classified by a single parameter $0 < \eta \le 2$ that characterizes the short-distance diffusive behavior of the unfolded energy levels $\overline{\Delta\epsilon_i^2} = D\Delta x^\eta$. Most physical systems correspond to the class $\eta = 2$ of differentiable GPs. The universality of all correlators under the level-diffusion scaling $x \to \bar{x} = D^{1/\eta}x$ emerges naturally in this framework and any correlator can be easily obtained from random-matrix simulations. Focusing on eigenfunction correlators, we have demonstrated their universality in chaotic and disordered systems as well as their sensitivity to time-reversal symmetry breaking.

ACKNOWLEDGMENT

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APPENDIX: THE SUPERSYMMETRY METHOD

Here we describe in some detail the calculations leading to the results (66) and (67) for the level density PC and (76) and (77) for the energy-dependent parametric overlap. We use the supersymmetry method invented by Efetov [12] (for a review see [28]). This method facilitates RMT calculations (e.g., PCs), which may be hard to perform using the orthogonal polynomials technique [2]. It is also non-perturbative and is therefore advantageous in comparison with the disorder perturbation theory [30] and the replica method [35].

In order to use the supersymmetry method the quantity to be calculated must be expressed in terms of Green functions. These are written as a Gaussian integral over a graded vector Ψ:

$$G(E^\pm) = \mp i \int d\Psi (SS^\dagger) \exp\left[\pm\frac{i}{2}\Psi^\dagger(E^\pm - H)\Psi\right] . \quad \text{(A1)}$$

If H is an N-dimensional matrix, Ψ is $4N$ dimensional and is defined in terms of the vector S of commuting variables and the vector χ of anticommuting variables, both of dimension N:

$$\Psi = (S, S^*, \chi, \chi^*) , \qquad \Psi^\dagger = (S^\dagger, S^T, \chi^\dagger, -\chi^T) . \quad \text{(A2)}$$

The integration measure in (A1) is

$$d\Psi = \prod_{i=1}^{N}\left[\frac{d(\mathrm{Re}S_i)d(\mathrm{Im}S_i)}{\pi} d\chi_i^* d\chi_i\right] . \quad \text{(A3)}$$

Correlators such as (65) and (75) require averaging products of two Green functions, namely, $A_k^{-\pm} = \overline{\mathrm{Tr}[G(E^-,x)]\mathrm{Tr}[G(E'^\pm,x')]}$ and $A_o^{-\pm} = \overline{\mathrm{Tr}[G(E^-,x)G(E'^\pm,x')]}$, respectively. We write

$$A_{k,o}^{-+} = \int d\Psi\, p_{k,0}(\Psi) \exp\left[\frac{i}{2}\Psi^\dagger\left(-\overline{E}\Lambda + \frac{\Omega}{2} + i\delta\right)\Psi\right] \times \overline{\exp\left(\frac{i}{2}\Psi^\dagger\Lambda H\Psi\right)} , \quad \text{(A4)}$$

where $\Psi = (\Psi_1, \Psi_2)$ (A2) and the matrices H and Λ are

$$H = \begin{pmatrix} H(x)\times I_4 & \\ & H(x')\times I_4 \end{pmatrix} ,$$
$$\Lambda = \begin{pmatrix} I_N \times I_4 & \\ & -I_N \times I_4 \end{pmatrix} . \quad \text{(A5)}$$

The notation $M \times I_n$ refers to the block diagonal matrix with n blocks M. Both Ψ, H, and Λ are of dimension $8N$. The preexponential factors are $p_k(\Psi) = (S_1^\dagger S_1)(S_2^\dagger S_2) = \frac{1}{4}(\Psi_1^{B\dagger}\Psi_1^B)(\Psi_2^{B\dagger}\Psi_2^B)$ and $p_o(\Psi) = (S_1^\dagger S_2)(S_2^\dagger S_1) = \frac{1}{4}(\Psi_1^{B\dagger}\Psi_2^B)(\Psi_2^{B\dagger}\Psi_1^B)$ with the superscript B referring to the commuting part of Ψ. We have also defined the energy mean $\overline{E} = (E+E')/2$ and difference $\Omega = E - E'$. Since the term in the exponent in (A4) is a Gaussian random variable, we can easily perform the GP average to obtain (with $a \to a/N$ to facilitate the steepest-descent treatment below)

$$\overline{\exp\left(\frac{i}{2}\Psi^\dagger\Lambda H\Psi\right)} = \exp\left[-\frac{1}{8}\overline{(\Psi^\dagger\Lambda H\Psi)^2}\right] = \exp\left(-\frac{a^2}{8N}\mathrm{Tr}g(\tilde{A}^2)\right) , \quad \text{(A6)}$$

where $\tilde{A}$ is a graded 8×8 matrix that is quadratic in the components of Ψ and depends on $f = f(x-x')$. It is convenient to separate out the f dependence by defining $\tilde{A}$ in terms of a graded 8×8 matrix A, given by

$$A = i\Lambda^{1/2}\Psi\otimes\Psi^\dagger\Lambda^{1/2} ; \quad \text{(A7)}$$

$\tilde{A}$ is then given by

$$\tilde{A} = \begin{pmatrix} A_{11} & \sqrt{f}A_{12} \\ \sqrt{f}A_{21} & A_{22}\end{pmatrix} = \frac{1+\sqrt{f}}{2}A + \frac{1-\sqrt{f}}{2}\Lambda A\Lambda , \quad \text{(A8)}$$

where the subscripts $1,2$ refer to Ψ_1,Ψ_2 in (A4). Using (A7) and (A8) it can be easily shown that $\Psi^\dagger\Psi = i\mathrm{Tr}g(\tilde{A}\Lambda)$ and (A4) can be rewritten as

$$A^{-+}_{k,o} = \int d\Psi\, p_{k,o}(\Psi)\exp\left(-\frac{i}{2}\overline{E}\Psi^\dagger\Lambda\Psi\right) \times\exp\left[-\frac{a^2}{8N}\mathrm{Tr}g(\tilde{A}^2) - \frac{1}{2}\left(\frac{\Omega}{2}+i\delta\right)\mathrm{Tr}g(\tilde{A}\Lambda)\right] . \quad \text{(A9)}$$

The integral representation (A4), which had a random component and a quadratic action, has now been transformed into (A9), which has no randomicity but a quartic action. Next, the quartic part is reduced to quadratic by a Hubbard-Stratonovich transformation

$$\exp\left[-\frac{a^2}{8N}\mathrm{Tr}g(\tilde{A}^2) - \frac{1}{2}\left(\frac{\Omega}{2}+i\delta\right)\mathrm{Tr}g(\tilde{A}\Lambda)\right] = \int dR\exp\left[-\frac{N}{2a^2}\mathrm{Tr}g(R^2) - \frac{i}{2}\mathrm{Tr}g(R\tilde{A}) - i\frac{N}{a^2}\left(\frac{\Omega}{2}+i\delta\right)\mathrm{Tr}g(R\Lambda)\right] , \quad \text{(A10)}$$

which holds in the limit $N\to\infty$ if $\delta\ll\Delta$. R is a graded 8×8 and consists of 16 commuting and 16 anticommuting independent variables and parametrizes the graded Lie group UOSP$(2,2/2,2)$ [see [28] for a discussion of the symmetry properties of R and the convergence issue, which motivates the third term in the exponent on the right-hand side of (A10)].

Using (A10) together with

$$\mathrm{Tr}g(R\tilde{A}) = \mathrm{Tr}g(\tilde{R}A) = -i\Psi^\dagger\Lambda^{1/2}\tilde{R}\Lambda^{1/2}\Psi , \quad \text{(A11)}$$

where

$$\tilde{R} = \frac{1+\sqrt{f}}{2}R + \frac{1-\sqrt{f}}{2}\Lambda R\Lambda \quad \text{(A12)}$$

[see (A8)], we can now write (A9) as

$$A^{-+}_{k,o} = \int dR\,\exp\left[-\frac{N}{2a^2}\mathrm{Tr}g(R^2) - i\frac{N}{a^2}\left(\frac{\Omega}{2}+i\delta\right)\mathrm{Tr}g(R\Lambda)\right]\int d\Psi\, p_{k,o}(\Psi)\exp\left[-\frac{1}{2}\Psi^\dagger\Lambda^{1/2}(\tilde{R}+i\overline{E})\Lambda^{1/2}\Psi\right] . \quad \text{(A13)}$$

Performing the Ψ integral exactly, we obtain

$$A^{-+}_{k,o} = \int dR\, p_{k,0}(R) \times\exp\left[-i\frac{N}{a^2}\left(\frac{\Omega}{2}+i\delta\right)\mathrm{Tr}g(R\Lambda) + \frac{1}{8}N\kappa|x-x'|^\eta\mathrm{Tr}g[(R+i\overline{E})^{-1}(\Lambda R\Lambda - R)]\right] \times\exp\left[-\frac{N}{2a^2}\mathrm{Tr}g(R^2) + \frac{N}{2}\mathrm{Tr}g[\ln(R+i\overline{E})]\right] , \quad \text{(A14)}$$

using (A12) and the short-distance expansion for f (51). The $8N$-dimensional integral expression (A9) has now been reduced to a 32-dimensional one. A steepest-descent treatment [28] gives

$$A^{+-}_{k,o} = \int dQ\, p_{k,o}(Q)\exp\left[-i\frac{\pi}{2\Delta}\left(\frac{\Omega}{2}+i\delta\right)\mathrm{Tr}g(Q\Lambda) + \frac{1}{16}N\kappa|x-x'|^\eta\mathrm{Tr}g([Q,\Lambda]^2)\right] , \quad \text{(A15)}$$

where Q is an 8×8 graded matrix consisting of 8 commuting and 8 anticommuting independent variables, which parametrizes the saddle-point manifold. The pre-exponential factors become $p_k(Q) = \frac{1}{4}\mathrm{Tr}Q^{BB}_{11}\mathrm{Tr}Q^{BB}_{22}$ and $p_o(Q) = \frac{1}{4}\mathrm{Tr}Q^{BB}_{21}\mathrm{Tr}Q^{BB}_{12}$, where the Ψ_1 and Ψ_2 bosonic and fermionic components of Q are arranged as in $\Psi\times\Psi^\dagger$.

In order to perform the Q integration it is convenient to use Efetov's parametrization [12], which has the advantage that the term in the exponent includes only commuting variables. Q is decomposed into $Q = UW\bar{U}$, where all the anticommuting variables $\eta_1,\eta_2,\rho_1,\rho_2$ are contained in U, which depends also on the commuting

variables $0 \leq \phi, \chi \leq \pi$, m real, and m_1 complex. The rest of the commuting variables $\theta_{1,2} \geq 0$ and $0 \leq \theta \leq \pi$ are contained in W (see [12] for the details, including the integration measure dQ). The preexponential factor in the orthogonal case is found to be

$$\begin{aligned} p_k(Q) &= -2^{12}(\cosh\theta_1 \cosh\theta_2 - \cos\theta)^2 \\ &\quad \times \eta_1^* \eta_1 \eta_2^* \eta_2 \rho_1^* \rho_1 \rho_2^* \rho_2 - 4\cos^2\theta \,, \\ p_o(Q) &= \left[2^{12} \sin^2(\theta) \left(\frac{1 - m^2 - |\, m_1 \,|^2}{1 + m^2 + |\, m_1 \,|^2} \right)^2 \right. \\ &\quad \left. + 2^{10}(\sinh^2\theta_1 \cosh^2\theta_2 + \cosh^2\theta_1 \sinh^2\theta_2) \right] \\ &\quad \times \eta_1^* \eta_1 \eta_2^* \eta_2 \rho_1^* \rho_1 \rho_2^* \rho_2 \\ &\quad + 4\sin^2(\theta) \left(\frac{1 - m^2 - |\, m_1 \,|^2}{1 + m^2 + |\, m_1 \,|^2} \right)^2 , \end{aligned} \tag{A16}$$

keeping only the terms of order 8 and 0 in the noncommuting variables since the others vanish upon integration. The integrations over all the variables are elementary except for $\lambda = \cos\theta$, $\lambda_1 = \cosh\theta_1$, and $\lambda_2 = \cosh\theta_2$ in terms of which the final results (66) and (76) are expressed. It is easy to show that

$$\begin{aligned} A_k^{--} &= \overline{\mathrm{Tr}[G(E^-, x)]\mathrm{Tr}[G(E'^-, x')]} \\ &= \overline{\mathrm{Tr}[G(E^-, x)]}\ \overline{\mathrm{Tr}[G(E'^-, x')]} \end{aligned} \tag{A17}$$

because there exists only a saddle point rather than a manifold, whereas

$$A_o^{--} = \overline{\mathrm{Tr}[G(E^-, x)G(E'^{\pm}, x')]} = 0 \tag{A18}$$

due to the vanishing of the preexponential factor $p_0(Q)$. A similar calculation using the appropriate parametrization for the unitary case [12] leads to the results (67) and (77).

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